

SIGN CHANGING SOLUTIONS FOR ELLIPTIC EQUATIONS WITH CRITICAL GROWTH IN CYLINDER TYPE DOMAINS

PEDRO GIRÃO AND MIGUEL RAMOS

ABSTRACT. We prove the existence of positive and of nodal solutions for $-\Delta u = |u|^{p-2}u + \mu|u|^{q-2}u$, $u \in H_0^1(\Omega)$, where $\mu > 0$ and $2 < q < p = 2N(N-2)$, for a class of open subsets Ω of \mathcal{R}^N lying between two infinite cylinders.

INTRODUCTION

We are concerned with the existence of nonzero solutions for the nonlinear second order elliptic equation

$$-\Delta u = |u|^{p-2}u + \mu|u|^{q-2}u, \quad u \in H_0^1(\Omega), \quad (\text{P})$$

where Ω is a smooth unbounded domain of \mathcal{R}^N with $N \geq 3$, $\mu \in \mathcal{R}^+$, $2 < q < p$ and p is the critical Sobolev exponent $p = 2^* = 2N/(N-2)$. Without loss of generality we assume that $0 \in \Omega$.

In the case where Ω is bounded, the proof of the existence of positive and of nodal (sign changing) solutions for (P) or similar equations goes back to the work in [3, 4, 10]. In the case where Ω is unbounded and p is subcritical ($p < 2^*$), we refer for example to [5, 12]. On the other hand, motivated by the work in [1, 2, 5, 7], in [8] the authors prove the existence of a positive solution for a class of unbounded domains, concerning the (somewhat simpler) equation $-\Delta u = \lambda u + |u|^{p-2}u$, where λ is positive and small (see also [9] for a related result).

The present work complements the quoted results. Following [5, 8], we fix a number $1 \leq \ell \leq N-1$ and write $\mathcal{R}^N = \mathcal{R}^\ell \times \mathcal{R}^{N-\ell}$, $z = (t, y) \in \mathcal{R}^\ell \times \mathcal{R}^{N-\ell}$. For a given subset $A \subset \mathcal{R}^{N-\ell}$ we denote $A_\delta = \{y \in \mathcal{R}^{N-\ell} : \text{dist}(y, A) < \delta\}$ and $\hat{A} = \mathcal{R}^\ell \times A$. Also, for $t \in \mathcal{R}^\ell$ we let $\Omega^t = \{y \in \mathcal{R}^{N-\ell} : (t, y) \in \Omega\}$. We shall consider both situations (H) and (H)₀ below:

- (H) There exist two nonempty bounded open sets $F \subset G \subset \mathcal{R}^{N-\ell}$ such that F is a Lipschitz domain and $\hat{F} \subset \Omega \subset \hat{G}$. Moreover, for each $\delta > 0$ there is $R > 0$ such that $\Omega^t \subset F_\delta$ for all $|t| \geq R$.
- (H)₀ There exists an open bounded set $G \subset \mathcal{R}^{N-\ell}$ such that $\Omega \subset \hat{G}$ and moreover for each $\delta > 0$ there is $R > 0$ such that $\Omega^t \subset B_{\mathcal{R}^{N-\ell}}(0, \delta)$ for all $|t| \geq R$.

2000 *Mathematics Subject Classification.* 35J20, 35J25, 35J65, 35B05.

Key words and phrases. Nodal solutions, cylindrical domains, semilinear elliptic equation, critical Sobolev exponent, concentration-compactness.

The authors are partially supported by FCT.

We have denoted by $B_{\mathcal{R}^{N-\ell}}(0, \delta)$ the open ball in $\mathcal{R}^{N-\ell}$ centered at the origin with radius $\delta > 0$. The case $(H)_0$ can be seen as a limit case of (H) , with $F = \{0\}$. We prove the following.

Theorem 1. *Consider problem (P) with $2 < q < p = 2^*$ and assume either (H) or $(H)_0$. Then, for every $\mu > 0$, the problem admits a positive (and a negative) solution of least energy.*

In order to prove the existence of nodal solutions in case (H) , we impose further restrictions on Ω , namely that Ω approaches \widehat{F} "smoothly and slowly."

$(H)'$ Assume (H) and that Ω is of class $C^{1,1}$ in such a way that the local charts as well as their inverses have uniformly bounded Lipschitz constants. Moreover, there exist constants $m > 0$ and $0 < a_1 < a_0$ such that $\left(1 + \frac{a}{|t|^m}\right) F \subset \Omega^t$ for every $a \in [a_1, a_0]$ and every $|t|$ large.

Theorem 2. *Consider problem (P) with $2 < q < p = 2^*$ and assume either $(H)'$ or $(H)_0$. In case $(H)_0$ holds, assume moreover that $q > (N+2)/(N-2)$. Then, for every $\mu > 0$, the problem admits a sign changing solution.*

In Theorem 2 the conclusion is that (P) has a pair of sign changing solutions, since the nonlinearity is odd. In case $(H)_0$, the extra restriction on q is merely needed in lower dimensions ($N = 3, 4, 5$), since $(N+2)/(N-2) \geq 2$ for $N \geq 6$. In fact, Theorem 2 still holds if $q = (N+2)/(N-2)$ provided μ is sufficiently large (see the remark which follows the proof of Proposition 2.5).

The proof of our main theorems is given in Section 2 (see Propositions 1.1 and 1.4); it relies on the concentration-compactness principle at infinity and on some ideas of [4, 8]. Section 3 provides technical estimates which are needed in the proof of Theorem 2. We also give further information on the decay properties of the solutions found in Theorems 1 and 2.

1. CONCENTRATION-COMPACTNESS

It is well known that the solutions of (P) correspond to critical points of the energy functional (for simplicity of notations, we take $\mu = 1$ in (P)):

$$I(u) = \frac{1}{2} \int |\nabla u|^2 - \frac{1}{p} \int |u|^p - \frac{1}{q} \int |u|^q, \quad u \in H_0^1(\Omega),$$

where the integrals are taken over the domain Ω . We recall $2 < q < p = 2^*$. It follows from assumptions (H) or $(H)_0$ that we can choose the norm $\|u\| := (\int |\nabla u|^2)^{1/2}$ in $H_0^1(\Omega)$. Let

$$c_0 := \inf\{I(u) : u \in H_0^1(\Omega), u \neq 0 \text{ and } I'(u)u = 0\}. \quad (1.1)$$

It is also clear that $c_0 > 0$ and that every nonzero critical point u of I is such that $I(u) \geq c_0$. The following result proves Theorem 1.

Proposition 1.1. *Under assumptions (H) or $(H)_0$, the infimum in (1.1) is attained in a critical point of I .*

Proof. 1. We shall omit what concerns standard arguments (cf. [3, 4]). We first recall that there exists a Palais-Smale sequence $(u_n) \subset H_0^1(\Omega)$ at level c_0 , namely

$$I(u_n) \rightarrow c_0 \quad \text{and} \quad I'(u_n) \rightarrow 0. \quad (1.2)$$

Since moreover $c_0 > 0$, (1.2) implies that $\liminf \|u_n\| > 0$. This sequence is bounded and, up to a subsequence, $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$, $u_n(x) \rightarrow u(x)$ a.e. and $I'(u) = 0$, $I(u) \geq 0$. Since $\liminf \|u_n\| > 0$ and $I'(u_n)u_n \rightarrow 0$, we also have that $\liminf \int |u_n|^p > 0$; indeed, if $\int |u_n|^p \rightarrow 0$ along a subsequence, then, since $(\int u_n^2)$ is bounded, by interpolation $\int |u_n|^q \rightarrow 0$, whence $\|u_n\| \rightarrow 0$, as $I'(u_n)u_n \rightarrow 0$.

2. Up to subsequences, there exist measures μ and ν on Ω such that $|\nabla(u_n - u)|^2 \rightharpoonup \mu$ and $|u_n - u|^p \rightharpoonup \nu$ weakly in the space $M(\Omega)$ of finite measures in Ω . Clearly, $\|\mu\| \geq S\|\nu\|^{2/p}$, where S is the best constant for the embedding $H^1(\mathcal{R}^N) \subset L^p(\mathcal{R}^N)$. By testing $I'(u_n) \rightarrow 0$ with $u_n\varphi$ for any $\varphi \in \mathcal{D}(\mathcal{R}^N)$ and since $I'(u)u\varphi = 0$ we also see that

$$\|\mu\| = \|\nu\|. \quad (1.3)$$

In particular,

$$\mu \neq 0 \Rightarrow \|\mu\| \geq S^{p/(p-2)} = S^{N/2}. \quad (1.4)$$

3. Define

$$\begin{aligned} \mu_\infty &:= \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| > R} |\nabla u_n|^2, \\ \nu_\infty &:= \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| > R} |u_n|^p, \\ \eta_\infty &:= \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| > R} |u_n|^q. \end{aligned}$$

Again, it is clear that

$$\mu_\infty \geq S \nu_\infty^{2/p}. \quad (1.5)$$

By testing $I'(u_n) \rightarrow 0$ with $u_n\psi_R$ ($R > 0$) where $\psi_R \in C^\infty(\Omega)$, $0 \leq \psi_R \leq 1$ is such that $\psi_R(x) = 0$ if $|x| \leq R$ and $\psi_R(x) = 1$ if $|x| \geq R+1$, it follows easily that

$$\mu_\infty = \nu_\infty + \eta_\infty. \quad (1.6)$$

4. We recall from [1, 2, 11] that

$$\begin{aligned} \int |\nabla u_n|^2 &= \int |\nabla u|^2 + \|\mu\| + \mu_\infty + o(1), \\ \int |u_n|^p &= \int |u|^p + \|\nu\| + \nu_\infty + o(1), \\ \int |u_n|^q &= \int |u|^q + \eta_\infty + o(1). \end{aligned}$$

As a consequence, and thanks to (1.2), (1.3) and (1.6), we have that

$$c_0 = I(u) + \left(\frac{1}{2} - \frac{1}{p}\right) \|\mu\| + \left(\frac{1}{2} - \frac{1}{p}\right) \nu_\infty + \left(\frac{1}{2} - \frac{1}{q}\right) \eta_\infty. \quad (1.7)$$

In particular, $c_0 \geq I(u)$. Since $I'(u) = 0$, the proof will be complete once we show that $u \neq 0$. Indeed, in this case we have that $I(u) \geq c_0$, whence $I(u) = c_0$. (Incidentally, (1.6) and (1.7) also show that, in fact, $\|\mu\| = \mu_\infty =$

0, hence $u_n \rightarrow u$ in $H_0^1(\Omega)$.)

5. We recall from [3] that $c_0 < S^{N/2}/N$. Since (1.7) implies that

$$c_0 \geq \left(\frac{1}{2} - \frac{1}{p}\right) \|\mu\| = \frac{1}{N} \|\mu\|,$$

we deduce from (1.3)-(1.4) that $\mu = \nu = 0$. Thus $u_n \rightarrow u$ in $H_{\text{loc}}^1(\Omega)$ and

$$c_0 = I(u) + \left(\frac{1}{2} - \frac{1}{p}\right) \nu_\infty + \left(\frac{1}{2} - \frac{1}{q}\right) \eta_\infty. \quad (1.8)$$

6. Suppose first that $\Omega = \widehat{F}$. Since $\liminf \int |u_n|^p > 0$, by Lemma 2.1 in [8] we may assume that, up to translations, $\int_{B_1(0)} |u_n|^p \geq c$ for some $c > 0$. Since $u_n \rightarrow u$ in $H_{\text{loc}}^1(\Omega)$, we conclude that $u \neq 0$ and this proves Proposition 1.1 for the case $\Omega = \widehat{F}$. Moreover, the argument shows that $c_0(\widehat{F}_\delta) \rightarrow c_0(\widehat{F})$ as $\delta \rightarrow 0$ (see (H) and (1.12) for the notations).

7. We complete the proof in case (H)₀ holds. Assume by contradiction that $u = 0$. Then, clearly $\int u_n^2 \rightarrow 0$ (see e.g. (2.1) in [8]). By interpolation, also $\int |u_n|^q \rightarrow 0$. In particular, $\eta_\infty = 0$. Since $c_0 < S^{N/2}/N$, (1.5), (1.6) and (1.8) show that then $\nu_\infty = 0$, whence, by the second identity in Step 4, $\int |u_n|^p \rightarrow 0$. This contradicts the fact that $\liminf \int |u_n|^p > 0$ and proves Proposition 1.1 under (H)₀.

8. At last, we consider the case where (H) holds and $\Omega \neq \widehat{F}$. Again, assume by contradiction that $u = 0$. Let $\delta > 0$ be given and take $R > 0$ according to assumption (H). Let ψ_R be as in Step 3 and denote

$$v_n = u_n \psi_R \in H_0^1(\widehat{F}_\delta).$$

Since $u_n \rightarrow 0$ in $H_{\text{loc}}^1(\Omega)$, clearly we have that

$$I(v_n) = I(u_n) + o(1) \quad \text{and} \quad I'(v_n)v_n = o(1). \quad (1.9)$$

We claim that

$$I(v_n) + o(1) \geq c_0(\widehat{F}_\delta). \quad (1.10)$$

Assuming the claim for a moment, it follows from (1.9)-(1.10) that

$$c_0 = I(u_n) + o(1) = I(v_n) + o(1) \geq c_0(\widehat{F}_\delta).$$

Since $\delta > 0$ is arbitrary, we conclude that $c_0 \geq c_0(\widehat{F})$. On the other hand, since $\widehat{F} \subset \Omega$ and $c_0(\widehat{F})$ is attained (see Step 6 above), we must have that $c_0 < c_0(\widehat{F})$. This contradiction completes the proof.

It remains to prove the inequality in (1.10). For this, we observe that (1.9) together with the fact that $\liminf I(u_n) > 0$ implies that $\liminf \|v_n\| > 0$ and $\liminf \int |v_n|^p > 0$. Now, let

$$w_n = t_n v_n \quad (t_n > 0)$$

be such that $I'(w_n)w_n = 0$; namely, t_n is given by

$$\frac{t_n^{p-2} \int |v_n|^p + t_n^{q-2} \int |v_n|^q}{\int |\nabla v_n|^2} = 1.$$

Then (t_n) is bounded and, since $I'(v_n)v_n \rightarrow 0$, we see that $t_n \rightarrow 1$. In particular,

$$I(w_n) = I(v_n) + o(1). \quad (1.11)$$

Now, by definition, $I(w_n) \geq c_0(\widehat{F}_\delta)$ and (1.10) follows from (1.11). \square

Using the notation in assumption (H), we denote

$$c_0(\widehat{F}) := \inf\{I(u) : u \in H_0^1(\widehat{F}), u \neq 0 \text{ and } I'(u)u = 0\} < S^{N/2}/N. \quad (1.12)$$

We also let

$$c_0^\infty := c_0(\widehat{F}) \quad \text{in case (H)}, \quad c_0^\infty := S^{N/2}/N \quad \text{in case (H)}_0. \quad (1.13)$$

We have shown in the proof of Proposition 1.1 that $c_0(\widehat{F})$ is attained by a critical point of the energy functional in $H_0^1(\widehat{F})$. In fact, the argument above yields the following compactness result.

Proposition 1.2. *Under assumptions (H) or (H)₀, let $(u_n) \subset H_0^1(\Omega)$ be such that*

$$\limsup I(u_n) < c_0^\infty \quad \text{and} \quad I'(u_n)(u_n\psi) \rightarrow 0 \quad (1.14)$$

for every $\psi \in C^\infty(\Omega) \cap W^{1,\infty}(\Omega)$. Suppose $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$, $u_n(x) \rightarrow u(x)$ a.e. and $I'(u)(u\psi) = 0$ for such functions ψ . Then $u_n \rightarrow u$ in $H_0^1(\Omega)$.

Proof. Since $I'(u)u = 0$, we have that $I(u) \geq 0$. Denote $v_n := u_n - u$. By the Brezis-Lieb Lemma,

$$I(v_n) = I(u_n) - I(u) + o(1) < c_0^\infty + o(1)$$

and

$$I'(v_n)(v_n\psi) = I'(u_n)(u_n\psi) - I'(u)(u\psi) + o(1) \rightarrow 0$$

for every $\psi \in C^\infty(\Omega) \cap W^{1,\infty}(\Omega)$. Since (v_n) converges weakly to zero, a similar (though easier) argument as in the proof of Proposition 1.1 shows that we cannot have $\limsup I(v_n) > 0$. Thus $I(v_n) \rightarrow 0$. Since also $I'(v_n)v_n \rightarrow 0$, we conclude that $\|v_n\| \rightarrow 0$, hence $u_n \rightarrow u$ in $H_0^1(\Omega)$. \square

Next we turn to the proof of Theorem 2. Following [4], let

$$c_1 := \inf\{I(u) : u \in H_0^1(\Omega), u^\pm \neq 0 \text{ and } I'(u^\pm)u^\pm = 0\} \geq c_0 > 0, \quad (1.15)$$

where we denote $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$. The following proposition will be proved in Section 3 (cf. Propositions 2.4 and 2.5).

Proposition 1.3. *Assume (H)' or (H)₀ holds; in the latter case, we also assume that $q > (N+2)/(N-2)$. Then*

$$c_1 < c_0 + c_0^\infty.$$

Our final result completes the proof of Theorem 2.

Proposition 1.4. *Assume (H)' or (H)₀ holds; in the latter case, we also assume that $q > (N+2)/(N-2)$. Then the infimum in (1.15) is attained in a critical point of I .*

Proof. It is known (cf. [4]) that there exists a Palais-Smale sequence at level c_1 , namely

$$I(u_n) \rightarrow c_1 \quad \text{and} \quad I'(u_n) \rightarrow 0,$$

with the additional property that

$$I(u_n^\pm) \geq c_0 + o(1) \quad (1.16)$$

(so that, in fact, $c_1 \geq 2c_0$). As in Step 1 in the proof of Proposition 1.1, modulo a subsequence, (u_n) converges weakly in $H_0^1(\Omega)$ and pointwise a.e. to a critical point u of I . Observe that $I'(u_n) \rightarrow 0$ implies that

$$I'(u_n^\pm)(u_n^\pm \psi) = I'(u_n)(u_n^\pm \psi) \rightarrow 0 \quad (1.17)$$

for every $\psi \in C^\infty(\Omega) \cap W^{1,\infty}(\Omega)$. Similarly, $I'(u^\pm)(u^\pm \psi) = 0$. Since moreover $I(u_n) = I(u_n^+) + I(u_n^-) = c_1 + o(1)$, we deduce from (1.16) and Proposition 1.3 that

$$\limsup I(u_n^\pm) < c_0^\infty. \quad (1.18)$$

It follows from (1.17), (1.18) and Proposition 1.2 that $u_n^\pm \rightarrow u^\pm$ in $H_0^1(\Omega)$. Hence $u_n \rightarrow u$ in $H_0^1(\Omega)$, $I(u) = c_1$ and $I(u^\pm) \geq c_0 > 0$. This finishes the proof. \square

2. DECAY AND ENERGY ESTIMATES

This section is devoted to general equations of the form

$$-\Delta u - \lambda u = g(u), \quad u \in H_0^1(\Omega), \quad (2.1)$$

where $\Omega \subset \mathcal{R}^N$ ($N \geq 3$) is an open set with $C^{1,1}$ boundary and g satisfies (recall that $p = 2^* = 2N/(N-2)$)

$$|g(s)| \leq C(|s| + |s|^{p-1}), \quad \forall s \in \mathcal{R}. \quad (2.2)$$

Under assumption (2.2), it follows from the Brezis-Kato estimates and classical elliptic regularity theory that the solutions of (2.1) lie in $C^2(\Omega) \cap L^\infty(\Omega) \cap C(\overline{\Omega})$. In view of the applications that we have in mind (cf. assumptions (H)-(H)₀), we let $\mathcal{R}^N = \mathcal{R}^\ell \times \mathcal{R}^{N-\ell}$ with $1 \leq \ell < N$ and accordingly write $(t, y) \in \mathcal{R}^\ell \times \mathcal{R}^{N-\ell}$ for any point $(t, y) \in \mathcal{R}^N$.

Proposition 2.1. *Let $\Omega = \mathcal{R}^\ell \times F$ where $F \subset \mathcal{R}^{N-\ell}$ is a $C^{1,1}$ domain and let $g \in C^1(\mathcal{R})$ satisfy (2.2), $g(0) = 0$ and $g'(s) = o(s^\varepsilon)$ near 0, for some $\varepsilon > 0$. Let u be a solution of*

$$-\Delta u - \lambda u = g(u), \quad u \in H_0^1(\Omega), \quad (2.3)$$

where $\lambda < \lambda_1$ and λ_1 is the first eigenvalue of $(-\Delta, H_0^1(F))$. Then

$$|u(t, y)| + |\nabla_t u(t, y)| \leq \varphi(y) e^{-\sqrt{1+(\lambda_1-\lambda)}|t|^2}, \quad \forall (t, y) \in \Omega, \quad (2.4)$$

where φ is a positive eigenfunction associated to λ_1 . Also, there exists a constant $C > 0$ such that

$$|\nabla u(t, y)| \leq C e^{-\sqrt{1+(\lambda_1-\lambda)}|t|^2}, \quad \forall (t, y) \in \Omega.$$

Proof. 1. Since $u \in L^\infty(\Omega)$, we have from (2.2) that $|g(u(x))| \leq c|u(x)|$ for every $x \in \Omega$. By elliptic regularity theory (Theorem 9.15 of [6]), there exists $c > 0$ such that, for all $\alpha \geq 2$,

$$\|u\|_{W^{2,\alpha}(B_1(0) \times F)} \leq c \|u\|_{L^\alpha(B_2(0) \times F)}.$$

Due to invariance by translations,

$$\|u\|_{W^{2,\alpha}(B_1(t) \times F)} \leq c \|u\|_{L^\alpha(B_2(t) \times F)} \quad \forall t \in \mathcal{R}^\ell. \quad (2.5)$$

In particular,

$$u(t, y) \rightarrow 0 \text{ as } |t| \rightarrow +\infty, \text{ uniformly for } y \in F \quad (2.6)$$

and

$$|\nabla u(t, y)| \rightarrow 0 \text{ as } |t| \rightarrow +\infty, \text{ uniformly for } y \in F. \quad (2.7)$$

2. Suppose $\mu \in]\lambda, \lambda_1[$ is fixed and let

$$\Psi(t) := \alpha e^{-\sqrt{1+(\lambda_1-\mu)|t|^2}} \in H^1(\mathcal{R}^\ell),$$

where α will be chosen later. An easy computation shows that

$$-\Delta \Psi + (\lambda_1 - \mu)\Psi = (\lambda_1 - \mu)\Psi((\ell - 1)\theta^{-1/2} + \theta^{-1} + \theta^{-3/2}) \quad (2.8)$$

where $\theta(t) := 1 + (\lambda_1 - \mu)|t|^2$. In particular,

$$-\Delta \Psi + (\lambda_1 - \mu)\Psi \geq \frac{\alpha(\lambda_1 - \mu)}{1 + (\lambda_1 - \mu)|t|^2} e^{-\sqrt{1+(\lambda_1-\mu)|t|^2}} =: h(t).$$

Let φ be a positive eigenfunction associated to λ_1 and

$$z(t, y) := \varphi(y)\Psi(t).$$

The function z satisfies

$$-\Delta z - \mu z \geq \varphi(y)h(t).$$

Hence, for $w := z - u$, we have

$$-\Delta w - \mu w \geq \varphi(y)h(t) + (\mu - \lambda)u - g(u) =: k(t, y). \quad (2.9)$$

Since $g(0) = 0 = g'(0)$, it follows from (2.6) that if $u(t, y) \geq 0$, then

$$(\mu - \lambda)u - g(u) \geq 0$$

if $|t| > R$, where R is chosen large; hence also $k(t, y) \geq 0$. In summary,

$$w < 0 \Rightarrow -\Delta w - \mu w \geq 0, \quad (2.10)$$

if $|t| > R$. Since $\partial z / \partial \nu = h \partial \varphi / \partial \nu < 0$ (ν stands for the outward normal to $\partial \Omega$), we can fix α so large that $w \geq 0$ for $|t| \leq R$. Let $\omega := \{x \in \Omega : w(x) < 0\}$. Since

$$w^-(x) = 0 \quad \forall x \in \partial \omega,$$

by multiplying (2.9) by w^- and integrating, it follows from (2.10) that $\omega = \emptyset$. Therefore $u \leq z$. In the same way we can prove that $-u \leq z$, and so

$$|u(t, y)| \leq \varphi(y) e^{-\sqrt{1+(\lambda_1-\mu)|t|^2}}, \quad \forall (t, y) \in \Omega; \quad (2.11)$$

the constant α has been incorporated into the function φ .

3. We now improve the previous estimate. Since $g'(s) = o(s^\varepsilon)$, there exists $C > 0$ such that

$$|g(u(t, y))| \leq C|u(t, y)|^{1+\varepsilon}, \quad \forall (t, y) \in \Omega. \quad (2.12)$$

We fix $\mu \in]\lambda, \lambda_1[$, sufficiently close to λ , so that

$$\gamma := (1 + \varepsilon)\sqrt{\lambda_1 - \mu} > \sqrt{\lambda_1 - \lambda}.$$

Combining (2.11) and (2.12),

$$|g(u(t, y))| \leq C\varphi(y)^{1+\varepsilon} e^{-\gamma|t|}, \quad \forall (t, y) \in \Omega. \quad (2.13)$$

Let $z(t, y) := \varphi(y)\Psi(t)$, where Ψ is like in Step 2, with μ replaced by λ . For $w := z - u$, we have

$$-\Delta w - \lambda w \geq \frac{\alpha(\lambda_1 - \lambda)}{1 + (\lambda_1 - \lambda)|t|^2} \varphi(y) e^{-\sqrt{1+(\lambda_1-\lambda)|t|^2}} - g(u(t, y)) =: p(t, y).$$

Since $\gamma > \sqrt{\lambda_1 - \lambda}$, it follows from (2.13) that $p(t, y) \geq 0$ if $|t|$ is large. Choosing α sufficiently large leads to $p \geq 0$ in Ω . We conclude from the maximum principle, as before, that $u \leq z$ in Ω and in the same way, $|u| \leq z$ in Ω .

4. To finish the proof we use the decay of u . Specifically, the derivatives $v = \partial u / \partial t_i$, for $i = 1, \dots, \ell$, satisfy

$$-\Delta v - \lambda v = g'(u)v \quad \text{and} \quad v \in H_0^1(\Omega).$$

The argument in Steps 2 and 3 above proves an analogous decay for v . The main point in the final argument is that if $\mu \in]\lambda, \lambda_1[$ is sufficiently close to λ then

$$\frac{\alpha(\lambda_1 - \lambda)}{1 + (\lambda_1 - \lambda)|t|^2} \varphi(y) e^{-\sqrt{1+(\lambda_1-\lambda)|t|^2}} - C \varphi^\varepsilon(y) e^{-\varepsilon \sqrt{1+(\lambda_1-\lambda)|t|^2}} \times \varphi(y) e^{-\sqrt{1+(\lambda_1-\mu)|t|^2}}$$

is positive for $|t|$ large. The final assertion in the statement of Proposition 2.1 follows from (2.5). \square

We now consider the setting analyzed in Section 2. Again, we denote by $\lambda_1 = \lambda_1(F)$ the first eigenvalue of $(-\Delta, H_0^1(F))$.

Proposition 2.2. *Suppose Ω is a domain satisfying assumption (H) and moreover that Ω is of class $C^{1,1}$ in such a way that the local charts as well as their inverses have uniformly bounded Lipschitz constants. Let $g \in C^1(\mathcal{R})$ be as in Proposition 2.1 and u be a solution of*

$$-\Delta u - \lambda u = g(u), \quad u \in H_0^1(\Omega),$$

with $\lambda < \lambda_1$. Then, for each $\bar{\lambda} \in]\lambda, \lambda_1[$, there exists a constant $C > 0$ such that

$$|u(t, y)| + |\nabla u(t, y)| \leq C e^{-\sqrt{1+(\bar{\lambda}-\lambda)|t|^2}}, \quad \forall (t, y) \in \Omega.$$

Proof. The proof is similar to that of Proposition 2.1, so we just stress the differences. Thanks to our assumption on Ω , the constant c in (2.5) can be taken uniformly bounded, hence (2.6) still holds. Now, fix $\delta > 0$ in such a way that $\lambda < \lambda_1(F_\delta) < \lambda_1$. Running through the argument in Step 2 of the proof of Proposition 2.1 we see that, similarly to (2.11),

$$|u(t, y)| \leq \varphi(y) e^{-\sqrt{1+(\lambda_1(F_\delta)-\mu)|t|^2}}, \quad \forall (t, y) \in \Omega, |t| \geq R,$$

provided $R > 0$ is sufficiently large; here, $\mu \in]\lambda, \lambda_1(F_\delta)[$ and φ is an eigenfunction associated to $\lambda_1(F_\delta)$. Arguing as in Step 3 of the quoted proof, the previous estimate for u can be improved to

$$|u(t, y)| \leq \varphi(y) e^{-\sqrt{1+(\lambda_1(F_\delta)-\lambda)|t|^2}}, \quad \forall (t, y) \in \Omega, |t| \geq R.$$

This clearly implies that we can choose $C > 0$ such that

$$|u(t, y)| \leq C e^{-\sqrt{1+(\lambda_1(F_\delta)-\lambda)|t|^2}}, \quad \forall (t, y) \in \Omega. \quad (2.14)$$

A similar decay estimate for the derivatives of u follows from (2.5) and (2.14). Since $\lambda_1(F_\delta)$ can be chosen arbitrarily close to λ_1 (see Lemma 2.3 of [8]), this proves the proposition. \square

Going back to Proposition 2.1, it may be interesting to observe that the asymptotic estimates can be sharpened as follows.

Proposition 2.3. *Under the assumptions of Proposition 2.1, let u be a solution of problem (2.3). Then:*

(a) *The conclusion of Proposition 2.1 still holds with $e^{-\sqrt{1+(\lambda_1-\lambda)|t|^2}}$ replaced by $e^{-\sqrt{1+(\lambda_1-\lambda)|t|^2}} |t|^{-\frac{\ell-1}{2}}$.*

(b) (Hopf lemma) *If u is positive and $\eta < \lambda$ then $u(t, y) \geq \tilde{\varphi}(y)e^{-\sqrt{1+(\lambda_1-\eta)|t|^2}}$ for every $(t, y) \in \Omega$, for some positive eigenfunction $\tilde{\varphi}$ associated to λ_1 .*

Proof. (a) We improve the estimate (2.4) by repeating the argument with

$$\Psi(t) := e^{-\sqrt{1+(\lambda_1-\lambda)|t|^2}} |t|^{-\frac{\ell-1}{2}}.$$

Indeed,

$$-\Delta\Psi + (\lambda_1 - \lambda)\Psi = \Psi \left((\lambda_1 - \lambda)\theta^{-1} + (\lambda_1 - \lambda)\theta^{-3/2} + \frac{\ell-1}{2} \frac{\ell-3}{2} \frac{1}{|t|^2} \right),$$

a computation that can be easily checked using (2.8); here, of course, $\theta(t) := 1 + (\lambda_1 - \lambda)|t|^2$. As a consequence, for sufficiently large $|t|$ we have that

$$-\Delta\Psi + (\lambda_1 - \lambda)\Psi \geq \frac{1}{2} e^{-\sqrt{1+(\lambda_1-\lambda)|t|^2}} |t|^{-\frac{\ell+3}{2}} =: h(t).$$

Due to the assumptions on g , for the function on $w := \alpha\varphi\Psi - u$, with α a fixed positive number, we have

$$-\Delta w - \lambda w \geq \alpha h(t)\varphi(y) - A\varphi^{1+\varepsilon}(y)e^{-(1+\varepsilon)\sqrt{1+(\lambda_1-\lambda)|t|^2}}.$$

The right hand member above is positive for sufficiently large $|t|$. Using the maximum principle, we conclude, as in (2.11), that

$$|u(t, y)| \leq \alpha\varphi(y)e^{-\sqrt{1+(\lambda_1-\lambda)|t|^2}} |t|^{-\frac{\ell-1}{2}}, \quad \forall (t, y) \in \Omega. \quad (2.15)$$

Finally, as in Step 4 of the quoted proof, a similar estimate for the derivatives of u follows from (2.4), (2.15) and the fact that

$$\begin{aligned} & \frac{\alpha}{2}\varphi(y)e^{-\sqrt{1+(\lambda_1-\lambda)|t|^2}} |t|^{-\frac{\ell+3}{2}} - \\ & - C\varphi^\varepsilon(y)e^{-\varepsilon\sqrt{1+(\lambda_1-\lambda)|t|^2}} |t|^{-\varepsilon\frac{\ell-1}{2}} \times \varphi(y)e^{-\sqrt{1+(\lambda_1-\lambda)|t|^2}} \end{aligned}$$

is positive for $|t|$ large.

(b) Here we let $\Psi(t) := e^{-\sqrt{1+(\lambda_1-\eta)|t|^2}}$. Fix any $\mu \in]\eta, \lambda[$. Similarly to (2.8), we can check that

$$h(t) := -\Delta\Psi + (\lambda_1 - \mu)\Psi \leq 0 \quad \text{for every } |t| \geq R$$

with R sufficiently large. Since $u(t, y) \rightarrow 0$ as $|t| \rightarrow \infty$ and since $g(0) = 0 = g'(0)$ we can choose R in such a way that also $(\mu - \lambda)u - g(u) \leq 0$ for $|t| \geq R$. Letting $z := \varphi\Psi$, we can fix a small $\alpha > 0$ so that $w := \alpha z - u \leq 0$ if $|t| \leq R$; this is possible because $u \in C^1(\overline{\Omega})$, $u > 0$ in Ω and $\partial u / \partial \nu < 0$ on $\partial\Omega$ (outward normal derivative). In summary, we have that (compare with (2.9))

$$-\Delta w - \mu w = \alpha\varphi h + (\mu - \lambda)u - g(u) =: k(t, y)$$

and $k(t, y) \leq 0$ for $|t| \geq R$, while $w \leq 0$ for $|t| \leq R$. Using the maximum principle as in the proof of Proposition 2.1 we conclude that $w \leq 0$ for all (t, y) . \square

We end this section with the proof of Proposition 1.3, which is contained in Propositions 2.4 and 2.5 below. We will refer to the functional I introduced at the beginning of Section 2 as well as to the quantities c_0 , c_0^∞ and c_1 defined in (1.1), (1.13) and (1.15), respectively.

Proposition 2.4. *Assume (H)' holds. Then $c_1 < c_0 + c_0^\infty$.*

Proof. 1. We know that c_0 is attained by a positive function $v \in H_0^1(\Omega)$ and c_0^∞ is attained by some positive function $\psi \in H_0^1(\mathcal{R}^\ell \times F)$ (cf. Proposition 1.1). Let $m > 0$ and $0 < a_1 < a_0$ be given by assumption (H)' and denote $A := a_0/a_1 > 1$. Fix a large number M such that $M > 2A$ and

$$\frac{a_1}{a_0} < \left(\frac{M-A}{M+A} \right)^m. \quad (2.16)$$

Let $\rho : \mathcal{R} \rightarrow \mathcal{R}$ be a smooth function such that $\rho(s) = 1$ for $|s| \leq 1$ and $\rho(s) = 0$ for $|s| \geq A$. We define ρ_R and η_R in \mathcal{R}^ℓ by $\rho_R = \rho(|t|/R)$ and $\eta_R(t) = \rho_R(t - MRe_1) = \rho(|\frac{t}{R} - Me_1|)$. We also let

$$v_R(t, y) := v(t, y)\rho_R(t)$$

and

$$\psi_R(t, y) := \lambda_R^{-N/p} \psi \left(\frac{t - MRe_1}{\lambda_R}, \frac{y}{\lambda_R} \right) \eta_R(t),$$

where

$$\lambda_R := 1 + \frac{a_0}{(M+A)^m R^m}. \quad (2.17)$$

We observe that v_R and ψ_R have disjoint supports. Moreover, both functions belong to $H_0^1(\Omega)$ if R is sufficiently large. Indeed, suppose $(t, y) \in \partial\Omega$ and let us show that $\psi_R(t, y) = 0$. We may already assume that $|t - MRe_1| \leq AR$. In particular,

$$(M-A)R \leq |t| \leq (M+A)R. \quad (2.18)$$

Now, to prove the claim it is sufficient to show that $(\frac{t - MRe_1}{\lambda_R}, \frac{y}{\lambda_R}) \notin \widehat{F}$, i.e. that $\frac{y}{\lambda_R} \notin F$. Observing that

$$y = \left(1 + \frac{a}{|t|^m} \right) \frac{y}{\lambda_R}$$

where, according to (2.16)-(2.18),

$$a := a_0 \left(\frac{|t|}{(M+A)R} \right)^m \in [a_1, a_0],$$

the conclusion follows from (H)' and the fact that $(t, y) \notin \Omega$.

2. Thanks to Proposition 2.2 (with $\lambda = 0$), we know that $|v(t, y)| + |\nabla v(t, y)| = O(e^{-\delta|t|})$ and similarly for ψ . Here and henceforth δ denotes various positive constants. It then follows easily that $I(v_R) \rightarrow I(v)$ and $I(\psi_R) \rightarrow I(\psi)$ as $R \rightarrow \infty$ and also that

$$I(v_R) = I(v) + O(e^{-\delta R}), \quad I(\psi_R) = I(\psi) + O(e^{-\delta R}). \quad (2.19)$$

In fact, the second estimate can be improved, observing that

$$\int \psi_R^p = \int \psi^p \rho_R^p = \int \psi^p + \int \psi^p (\rho_R^p - 1) = \int \psi^p + O(e^{-\delta R})$$

and similarly $\int |\nabla \psi_R|^2 = \int |\nabla \psi|^2 + O(e^{-\delta R})$, while

$$\int \psi_R^q = \lambda_R^{N(1-\frac{q}{p})} \int \psi^q + O(e^{-\delta R})$$

so that

$$\begin{aligned} I(\psi_R) &= I(\psi) + \left(1 - \lambda_R^{N(1-\frac{q}{p})}\right) \int \psi^q + O(e^{-\delta R}) \\ &\leq I(\psi) - N \left(1 - \frac{q}{p}\right) \frac{a_0}{(M+A)^m R^m} \int \psi^q + O(e^{-\delta R}), \end{aligned}$$

whence, for every sufficiently large R ,

$$I(\psi_R) < I(\psi). \quad (2.20)$$

3. Clearly, as in (2.19)-(2.20), for large R and uniformly for $\tau_1, \tau_2 \in [1/2, 2]$, we have that

$$\begin{aligned} I(\tau_1 v_R - \tau_2 \psi_R) &= I(\tau_1 v_R) + I(\tau_2 \psi_R) < I(\tau_1 v) + I(\tau_2 \psi) \\ &\leq \sup_{s \geq 0} I(sv) + \sup_{s \geq 0} I(s\psi) = c_0 + c_0^\infty. \end{aligned}$$

The last equality above is a direct consequence of the definitions of c_0 and c_0^∞ , by standard arguments (cf. [3, 4, 11]). In summary, there exists R_0 such that

$$\sup_{1/2 \leq \tau_1, \tau_2 \leq 2} I(\tau_1 v_R - \tau_2 \psi_R) < c_0 + c_0^\infty, \quad \forall R \geq R_0. \quad (2.21)$$

4. Thanks to (2.21), to complete the proof it remains to show that there exist $\tau_1, \tau_2 \in [1/2, 2]$ and $R \geq R_0$ such that $w := \tau_1 v_R - \tau_2 \psi_R$ satisfies $I'(w^\pm)w^\pm = 0$. Since v_R and ψ_R have disjoint supports, this amounts to prove that there exist $\tau_1, \tau_2 \in [1/2, 2]$ and $R \geq R_0$ such that

$$I'(\tau_1 v_R)v_R = 0 \quad \text{and} \quad I'(\tau_2 \psi_R)\psi_R = 0. \quad (2.22)$$

Now, we have that $I'(v_R/2)v_R \rightarrow I'(v/2)v > 0$ and $I'(2v_R)v_R \rightarrow I'(2v)v < 0$ as $R \rightarrow \infty$ and similarly for ψ . Hence (2.22) follows by applying the intermediate value theorem. \square

Proposition 2.5. *Assume (H)₀ holds and moreover that $q > (N+2)/(N-2)$. Then $c_1 < c_0 + c_0^\infty = c_0 + S^{N/2}/N$.*

Proof. Let $U(x) = c_N/(1+|x|^2)^{(N-2)/2}$ be the Talenti instanton, normalized in such a way that $\int |U|^p = \int |\nabla U|^2 = S^{N/2}$ (i.e. $c_N = (N(N-2))^{(N-2)/4}$). Let $U_\varepsilon(x) = \varepsilon^{-N/p}U(x/\varepsilon)$ be its rescaling, so that also $\int |U_\varepsilon|^p = \int |\nabla U_\varepsilon|^2 = S^{N/2}$. The following argument is similar to that in [12], except that we cut down the least energy solution and also U_ε and estimate the error in doing so, instead of computing the interference between their energies.

Recall that, without loss of generality, we are assuming that $0 \in \Omega$. By Proposition 1.1, we know that c_0 is achieved by a positive function $v \in H_0^1(\Omega) \cap C^1(\Omega)$. Let $\rho, \eta : \mathcal{R} \rightarrow \mathcal{R}$ be smooth functions such that $\rho(s) = 1$ for $|s| \leq 1$, $\rho(s) = 0$ for $|s| \geq 2$, $\eta(s) = 0$ for $|s| \leq 2$ and $\eta(s) = 1$ for $|s| \geq 3$. We define ρ_ε and $\eta_\varepsilon : \mathcal{R}^N \rightarrow \mathcal{R}$ by $\rho_\varepsilon(x) = \rho(|x|/\sqrt{\varepsilon})$ and $\eta_\varepsilon(x) = \eta(|x|/\sqrt{\varepsilon})$. We also define

$$u_\varepsilon := U_\varepsilon \rho_\varepsilon \quad \text{and} \quad v_\varepsilon := v \eta_\varepsilon.$$

It is clear that u_ε and v_ε have disjoint supports and that they both belong to $H_0^1(\Omega)$. We can estimate

$$\begin{aligned} \int |\nabla v_\varepsilon|^2 &\leq \int |\nabla v|^2 + 2 \left(\int_{2\varepsilon^{1/2} \leq |x| \leq 3\varepsilon^{1/2}} (|\nabla v|^2 \eta_\varepsilon^2 + v^2 |\nabla \eta_\varepsilon|^2) \right) \\ &\leq \int |\nabla v|^2 + O(\varepsilon^{N/2}) + O(\varepsilon^{(N-2)/2}) \\ &= \int |\nabla v|^2 + O(\varepsilon^{(N-2)/2}), \end{aligned}$$

while

$$\int v_\varepsilon^p = \int v^p + \int v^p (\eta_\varepsilon^p - 1) \geq \int v^p - \int_{|x| \leq 3\varepsilon^{1/2}} v^p \geq \int v^p + O(\varepsilon^{N/2})$$

and similarly for $\int v_\varepsilon^q$, so that

$$I(v_\varepsilon) \leq I(v) + O(\varepsilon^{(N-2)/2}). \quad (2.23)$$

As for u_ε ,

$$\begin{aligned} \int |\nabla u_\varepsilon|^2 &\leq \int |\nabla U_\varepsilon|^2 + 2 \left(\int |\nabla U_\varepsilon|^2 \rho_\varepsilon^2 + U_\varepsilon^2 |\nabla \rho_\varepsilon|^2 \right) \\ &\leq S^{N/2} + O(\varepsilon^{(N-2)/2}), \end{aligned}$$

while, denoting by $c > 0$ some constant which is independent of ε ,

$$\int u_\varepsilon^p \geq S^{N/2} + O(\varepsilon^{N/2}) \quad \text{and} \quad \int u_\varepsilon^q \geq c \varepsilon^{N(1-\frac{q}{p})},$$

as can be checked directly, using the explicit expression of U_ε . In summary,

$$I(u_\varepsilon) \leq \left(\frac{1}{2} - \frac{1}{p} \right) S^{N/2} + O(\varepsilon^{\frac{N-2}{2}}) - c \varepsilon^{N(1-\frac{q}{p})}. \quad (2.24)$$

Combining (2.23) and (2.24) yields

$$I(u_\varepsilon) + I(v_\varepsilon) \leq c_0 + \frac{S^{N/2}}{N} + c_1 \varepsilon^{\frac{N-2}{2}} - c_2 \varepsilon^{N(1-\frac{q}{p})}, \quad (2.25)$$

for some positive constants c_1 and c_2 . In particular,

$$I(u_\varepsilon) + I(v_\varepsilon) < c_0 + \frac{S^{N/2}}{N} \quad (2.26)$$

if ε is sufficiently small since, by assumption, $\frac{N-2}{2} > N(1-\frac{q}{p})$; indeed, this condition is equivalent to $q > p-1 = (N+2)/(N-2)$. From (2.26) we can end the proof of Proposition 2.5 with similar arguments as in Steps 3 and 4 in the proof of Proposition 2.4. \square

Remark 2.6. As observed at the beginning of Section 2, for simplicity of notations we have assumed that $\mu = 1$ in problem (P). In the general case, (2.25) reads as

$$I(u_\varepsilon) + I(v_\varepsilon) \leq c_0 + \frac{S^{N/2}}{N} + c_1 \varepsilon^{\frac{N-2}{2}} - \mu c_2 \varepsilon^{N(1-\frac{q}{p})}.$$

Thus one still has (2.26) in case $q = (N+2)/(N-2)$ provided μ is sufficiently large.

REFERENCES

- [1] A.K. Ben-Naoum, C. Troestler and M. Willem, Extrema problems with critical Sobolev exponents on unbounded domains. *Nonlinear Anal. TMA* **26** (1996) 823–833.
- [2] G. Bianchi, J. Chabrowski and A. Szulkin, On symmetric solutions of an elliptic equation with a nonlinearity involving critical Sobolev exponent. *Nonlinear Anal. TMA* **25** (1995) 41–59.
- [3] H. Brezis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. *Comm. Pure Appl. Math.* **36** (1983) 437–476.
- [4] G. Cerami, S. Solimini and M. Struwe, Some existence results for superlinear elliptic boundary value problems involving critical exponents. *J. Funct. Anal.* **69** (1986) 289–306.
- [5] M. Del Pino and P. Felmer, Least energy solutions for elliptic equations in unbounded domains. *Proc. Royal Soc. Edinburgh* **126A** (1996) 195–208.
- [6] D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*, Second edition. Grundlehren der Mathematischen Wissenschaften Vol. 224, Springer, New York (1983).
- [7] P.L. Lions, The concentration-compactness principle in the Calculus of Variations. The limit case, part 2. *Revista Matemática Iberoamericana* **1** (1985) 45–121.
- [8] M. Ramos, Z.-Q. Wang and M. Willem, Positive solutions for elliptic equations with critical growth in unbounded domains, in *Calculus of Variations and Differential Equations*, A. Ioffe, S. Reich and I. Shafrir Eds., Research Notes in Mathematics Series Vol. 140, Chapman & Hall/CRC, Boca Raton, FL (2000) 192–199.
- [9] I. Schindler and K. Tintarev, Abstract concentration compactness and elliptic equations on unbounded domains, in *Prog. Nonlinear Differential Equations Appl.* Vol. 43, M.R. Grossinho, M. Ramos, C. Rebelo and L. Sanchez Eds., Birkhäuser, Boston (2001) 369–380.
- [10] G. Tarantello, Nodal solutions of semilinear elliptic equations with critical exponent. *Differential and Integral Equations* **5** (1992) 25–42.
- [11] M. Willem, *Minimax theorems*. Prog. Nonlinear Differential Equations Appl. Vol. 24, Birkhäuser, Boston (1996).
- [12] X.-P. Zhu, Multiple entire solutions of a semilinear elliptic equations. *Nonlinear Analysis TMA* **12** (1998) 1297–1316.

MATHEMATICS DEPARTMENT, INSTITUTO SUPERIOR TÉCNICO, AV. ROVISCO PAIS,
1049-001 LISBOA, PORTUGAL

E-mail address: `pgirao@math.ist.utl.pt`

CMAF AND FACULTY OF SCIENCES, UNIVERSIDADE DE LISBOA, AV. PROF. GAMA
PINTO, 2, 1649-003 LISBOA, PORTUGAL

E-mail address: `mrmos@lmc.fc.ul.pt`